

MATHELANG
ENRICHMENT MINI BOOK
SEARCH FOR MEANING
THROUGH
P A T T E R N

2001

RAMANUJAN MUSEUM & MATH EDUCATION CENTRE
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BACKDROP

Within the first five years of life, most children have already learned to construct sentences and where the various kinds of words have to come in a sentence. The child feels its way to these rules unconsciously as a kind of habit and having acquired them can express its thoughts, can argue and think.

So it is with mathematics, and any one who can learn to speak and to express his thoughts through marks on a piece of paper can equally learn to do mathematics. So there is no special ability here.

..... methods used by many good teachers – methods of visualizing, dramatizing and analyzing numbers so that the attention and understanding of children can be gained and held

Prof. H. Levy

Dedicated

to

**the off beat Professor
W W SAWYER**

Whose exposition through his books
have had immense influence on my
growth of professionalism
as
a math teacher and educator
since the early fifties

As Author.

- * *Mathematician's Delight* (Pelican, 1943)
- * *A Path to Modern Mathematics* (Pelican, 1966)
- * *Concrete Approach to Abstract Algebra* (W H Freeman, 1959)
- * *An Engineering Approach to Linear Algebra* (CUP, 1972)

- * *Prelude to Mathematics* (Pelican, 1955)
- * *Vision in Elementary Mathematics* (Pelican, 1964)
- * *The Search for Pattern* (Pelican, 1970)

As Editor

- * *Mathematics in Theory and Practice*

- * *Mathematics Student Journal*

MATHELANG RESIDENTIAL SCHOOL

John and Ravi are in Primary V. Their Math Teacher believes in uncovering the prescribed syllabus and leaves a lot to be done by learners through experimentation and exploration, detection of emerging patterns and pushing ahead to break new ground in their self learning efforts.

As a mathelang teacher, he encourages children to raise questions arising out of curiosity and try to find the answers themselves if possible. If tackling the questions need extra study, the question raising children are allowed to visit the school library which has well stocked math section kept updated through annual addition of books and journals, videos and CD Roms.

What the teacher does is to help them communicate properly. That is all. Outside the classes, class distinctions are forgotten and children mix in the lawns. John and Ravi were happy when Salim from class IV joined them with his own discoveries.

What are apparently meaningless become meaningful through the study of pattern underlying their extended behaviour seen through analysis of signs or symbols.

The two have to their credit many anticipatory discoveries, which are covered in the books at different levels. They joined the programmes of morning assembly talk once a week by students and won appreciation from staff and envy from other students.

May this mini enrichment book trigger more such math loving children.

Acknowledgement: *Special thanks to Mr. Sadagopan Rajesh for his assistance in producing this fourth booklet.*

Author

Passage from Repeated Addition to Repeated Multiplication

Math club meet dialogues 1

Teacher:

$$\begin{array}{l} 7 + 7 + 7 = 3 \times 7 \\ 11 + 11 + 11 = 3 \times 11 \\ 26 + 26 + 26 = 3 \times 26 \end{array}$$

John:

$$a + a + a = 3a \quad (\text{meaning } 3 \text{ times } a)$$

Sir, what about $7 \times 7 \times 7$, $11 \times 11 \times 11$, $26 \times 26 \times 26$, etc.?

Teacher: Mathematicians write

$$7 \times 7 \times 7 = 7^3 \quad (\text{read } 7 \text{ to the power of } 3 \text{ or } 7 \text{ cubed})$$

$$\begin{array}{l} 11 \times 11 \times 11 = 11^3 \\ 26 \times 26 \times 26 = 26^3 \end{array}$$

John:

$$aaa = a^3$$

Teacher: 3 is called the *index*, *a* is the *base*
 a^3 is a power of *a*

Teacher:

$$\begin{array}{l} (5+5+5+5) + (5+5+5) \\ = 4 \times 5 + 3 \times 5 \\ = (4+3).5 \\ (8+8+8+8) + (8+8+8) \\ = 4 \times 8 + 3 \times 8 \\ = (4+3).8 \\ (10+10+10+10) + (10+10+10) \\ = 4 \times 10 + 3 \times 10 \\ = (4+3).10 \end{array}$$

John:

$$\begin{array}{l} (p+p+p+p) + (p+p+p) \\ = 4 \times p + 3 \times p = (4+3).p \end{array}$$

Ravi: I have done this kind of thing for repeated multiplication.

$$pppp \times ppp = p^4 \cdot p^3 = p^{4+3}$$

Am I right!

Teacher: Congrats.

(4)

Common Index Pattern

Math club meet dialogues 2

Teacher: What property is exhibited by the pattern here?

$$6 \times 6 \times 6 \times 5 \times 5 \times 5 = 6^3 \times 5^3$$

$$9 \times 9 \times 9 \times 4 \times 4 \times 4 = 9^3 \times 4^3$$

$$2 \times 2 \times 2 \times 7 \times 7 \times 7 = 2^3 \times 7^3$$

Ravi: $ddd \times ggg = d^3 \times g^3$

John: Can we not view the pattern this way?

$$(6 \times 5) \times (6 \times 5) \times (6 \times 5) = (6 \times 5)^3$$

$$(9 \times 4) \times (9 \times 4) \times (9 \times 4) = (9 \times 4)^3$$

$$(2 \times 7) \times (2 \times 7) \times (2 \times 7) = (2 \times 7)^3$$

This gets us then a new pattern.

$$6^3 \times 5^3 = (6 \times 5)^3$$

$$9^3 \times 4^3 = (9 \times 4)^3$$

$$2^3 \times 7^3 = (2 \times 7)^3$$

$$a^3 \times b^3 = (ab)^3$$

Laws of Indices through Pattern

Math club meet dialogues 3

Teacher: We shall now give in *Pattern Language* the laws of indices (i) when the *base* is *common*

(ii) when the *index* is *common*. What does 5^3 mean? The base 5 is repeated 3 times as a factor and 3 is the index. 5^3 is a power of 5.

$5^3 \times 5^4 = 5^{3+4}$: The common base 5 here is repeated in multiplication 3 times followed by repetition of the same base 5 four times, amounting to the total repetition of the base (3 + 4) times. Here indices are different but the base is the same.

$5^3 \times 7^3 = (5 \times 7)^3$: The base 5 is repeated in multiplication 3 times followed by the repetition of 7 the same number of times amounting to repetition of (5 × 7) three times. 3 becomes the common index.

Here bases are different, but the index is common.

Discover the *Common base law of indices*

$$6^4 \times 6^2 = 6^{4+2}$$

$$15^3 \times 15^7 = 15^{3+7}$$

$$8^5 \times 8^4 = 8^{5+4}$$

John: $a^m \times a^n = a^{m+n}$

Discover the *Common index law of indices*

$$9^3 \times 8^3 = (9 \times 8)^3$$

$$6^5 \times 2^5 = (6 \times 2)^5$$

$$17^2 \times 13^2 = (17 \times 13)^2$$

$$a^m \times b^m = (ab)^m$$

Salim said, 'these laws relate to multiplication. What about laws relate to division?'

Teacher said, 'I leave that for the members to discover.'

The trio fixed their findings on the vulcro board.

Finding by John & Ravi: $(8 \times 8 \times 8 \times 8 \times 8) \div (8 \times 8 \times 8) = 8^5 \div 8^3 = 8^{5-3}$

$$(7 \times 7 \times 7) \div (7 \times 7) = 7^3 \div 7^2 = 7^{3-2}$$

$$a^m \div a^n = a^{m-n}$$

Salim's finding: $(8 \times 8 \times 8 \times 8 \times 8 \times 8) = 8^6 = (8^2)^3$

$$(a^m)^n = a^{mn}$$

Meaning of Zero Index

The trio put their heads together to review the index laws discovered in the club meet. They discovered *two* more patterns in index laws.

$$(9 \times 9 \times 9) \div (9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9) = 1 \div (9 \times 9 \times 9 \times 9)$$

 $\Rightarrow$

$$9^3 \div 9^7 = 1 \div 9^{7-3}$$

$$(2 \times 2 \times 2 \times 2) \div (2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) = 1 \div (2 \times 2 \times 2)$$

 $\Rightarrow$

$$2^4 \div 2^6 = 1 \div 2^{6-4}$$

$$\text{P.L. } a^m \div a^n = 1 \div a^{n-m}$$

$$(4 \times 4 \times 4 \times 4) \div (2 \times 2 \times 2 \times 2) = (4 \div 2)^4$$

$$(5 \times 5 \times 5) \times (5 \times 5 \times 5) = 5^3 \times 5^3 = (5^3)^2 = 5^6$$

$$(6 \times 6 \times 6 \times 6 \times 6) \div (2 \times 2 \times 2 \times 2 \times 2) = (6 \div 2)^5$$

$$(5 \times 5) \times (5 \times 5) \times (5 \times 5) = 5^2 \times 5^2 \times 5^2 = (5^2)^3 = 5^6$$

P.L.

$$a^m \div b^m = (a \div b)^m$$

$$(a^m)^n = (a^n)^m$$

Salim asked, 'Can we have 0 as index number?' John and Ravi said, 'We shall explore.' Salim said, 'When will one get say 5^0 ?' John said, 'This will arise only in division. So let us take $5 \times 5 \times 5 = 5 \times 5 \times 5$. It means $5^3 \div 5^3 = 5^{3-3}$. But $5^3 \div 5^3 = 1$. So $5^0 = 1$. Ravi intervened and said, 'It can be got in another way also. $5^3 \div 5^3 = (5 \div 5)^3 = 1^3 = 1$. So $5^0 = 1$. Salim summed up by saying that in **P.L.** $a^m \div a^m = a^{m-m} = (a \div a)^m$ giving $a^0 = 1$ where $a \neq 0$. They become curious about practising it. They met their master with their enquiry. The master said that through paper folding, it could be visualised.

Take a sheet of paper. If it is folded once, 2 pieces can be got. If it is folded again, that is folded twice, 4 pieces are got. John could see the thrust. 1 folding 2 pieces, 2 folding 4 pieces, 3 folding 8 pieces and so on.

Then *no folding or zero folding, one piece or the whole*. So $2^0 = 1$. Salim asked, 'What about 3^0 and 4^0 etc.?' The teacher said, 'Just as there are 3 bed room flats and 4 bed room flats, we should take 3 part folding as one folding as only folding to get 3^0 and 4 part folding as 1 folding to get 4^0 .'

Meaning of negative index number

The trio got curious about the index number being negative. The bell rang. They went to their desks and tried to find the meaning. They met after breakfast and compared their notes.

John's Jotting

$$\begin{array}{ccccccc} & & & a^0 & a^1 & a^2 & a^3 \dots\dots \\ \dots\dots & a^{-3} & a^{-2} & a^{-1} & & & \\ & & & \frac{a^1}{a} & \frac{a^2}{a} & \frac{a^3}{a} & \\ & & & = & 1 & & \\ & \frac{1}{a^3} & \frac{1}{a^2} & \frac{1}{a} & & & \end{array}$$

Ravi's Jotting

$$a^{-3} \cdot a^3 = a^0 = 1$$
$$\therefore a^{-3} = \frac{1}{a^3}$$
$$\text{Or } a^3 = \frac{1}{a^{-3}}$$

Salim's Jotting

$$\begin{array}{ll} 5^4 = \frac{5^5}{5} & 5^1 = \frac{5^2}{5} \\ 5^3 = \frac{5^4}{5} & 5^0 = \frac{5^1}{5} = 1 \\ 5^2 = \frac{5^3}{5} & 5^{-1} = \frac{1}{5} \end{array}$$

After discussion, they all agreed to say, a^{-n} means reciprocal of a^n and a^n means reciprocal of a^{-n} .

Ravi was up with the question about *fractional index*. John tried and could not see his way. Salim kept watching and wondering how to start the exploration.

It was lunch time. They went to the dining hall. They saw the math master entering. They accosted him about their enquiry about fractional index and requested him to give a hint for getting started. The teacher said, 'Write down $\sqrt[4]{1}$, $\sqrt[4]{2}$, $\sqrt[4]{3}$, $\sqrt[4]{4}$ and so on and see the emerging pattern of indices.'

Soon after lunch, they went to their rooms. After an hour's rest time, they started working on the question and could see light at the end of the tunnel.

Meaning of fractional index (1)

The trio started working separately and decided to meet after breakfast the next day.

The breakfast was over and sat on the steps and took out their jottings and circulated them.

Ravi's Jotting

$$\sqrt{4^1} = 2$$

$$\sqrt{4^2} = 4^1 = 4^{2/2}$$

$$\sqrt{4^3} = \sqrt{4^2} \times \sqrt{4} = 4 \times 2$$

$$\sqrt{4^4} = 4^2 = 4^{4/2}$$

$$\text{So } \sqrt{4^1} = 2 = 4^{1/2}; \sqrt{4^3} = 4 \times 2 = 4^{2/2} \cdot 4^{1/2} \text{ and so on.}$$

$$\text{P.L. } \sqrt{a^p} = a^{p/2}$$

John's Jotting

$$\sqrt{4^2} = 4 = 4^{2/2}$$

$$\sqrt{4^4} = 4^2 = 4^{4/2}$$

$$\sqrt{4^6} = 4^3 = 4^{6/2}$$

$$\sqrt{4^8} = 4^4 = 4^{8/2}$$

$$\text{So } \sqrt{4} = 4^1 = 4^{1/2}; \sqrt{4^3} = 4^{3/2}$$

$$\sqrt{4^5} = 4^{5/2} \text{ and so on.}$$

$$\text{P.L. } \sqrt{n^k} = n^{k/2}$$

Salim's Jotting

$$\sqrt{4} = \sqrt{2 \times 2} = 2$$

$$\sqrt{4^2} = \sqrt{4 \times 4} = 4$$

$$\sqrt{4^4} = \sqrt{4 \times 4 \times 4 \times 4} = 4^2$$

So

$$\sqrt{4^6} = \sqrt{4 \times 4 \times 4 \times 4 \times 4 \times 4} = 4^3$$

$$\sqrt{4^4} = 4^2 = 4^{4/2}$$

$$\sqrt{4^6} = 4^3 = 4^{6/2}$$

They discovered that finding square root is tied with fractional index. It struck John to give a nice interpretation of the situation.

$4^{1/2}$ means 'splitting the base number 4 with *two equal factors* and taking *one* of them. $4^{3/2}$ means 'splitting the base number 4 with *two equal factors* and taking *one* of them *thrice*.

'What about $4^{-1/2}$ and $4^{-3/2}$?', Ravi asked. John replied, 'Why, it should be the reciprocal of $\frac{1}{4^{1/2}}$ and $\frac{1}{4^{3/2}}$ respectively. They walked up to the master having the evening stock and told him about their findings and interpretation. The master said, 'I am proud'.

Meaning of fractional index (2)

Having discovered the meaning of $4^{1/2}$ and $4^{3/2}$, they became curious about taking other fractional indices. After an informal chat while walking to their house after breakfast, they decided to prepare a combined paper for being put up on dining hall bulletin board. Salim volunteered to prepare the paper. John and Ravi agreed.

$8^{1/3} = 2$ as $8 = 2 \times 2 \times 2$ and one of the three equal factors is alone to be taken.

$8^{2/3} = 4$ as two of the three equal factors give $2 \times 2 = 4$

$16^{1/4} = 2$ as $16 = 2 \times 2 \times 2 \times 2$ and one of the equal factors is 2 ; $16^{3/4} = 2 \times 2 \times 2 = 8$ give $2 \times 2 = 4$ (three equal factors taken)

$16^{5/4} = 32$ as $16^{5/4} = 16^1 \times 16^{1/4} = 16 \times 2 = 32$

$243^{1/5} = 81$, $243^{6/5} = 729$ etc.

$1000000^{1/6} = 10$, $1000000^{1/2} = 1000$, $1000000^{1/3} = 100$, $10^{4/3} = 10 \times 10^{1/3}$ and so on.

In P.L. if $a^{1/p} = b$, then $a = b^p$ and

if $a^{c/d} = b$, then $a^c = b^d$, also $a = b^{d/c}$

John and Ravi went through Salim's paper and okayed it for being shown to the math master and put up on the bulletin board with his approval and comments. John suggested drawing up a repeated multiplication table and discovering its wonders.

Repeated Multiplication Table and its Wonders

The trio were so fascinated with their findings from the repeated multiplication table relating to base 2 that they told their master about their readiness to present a trilogy on it during the forthcoming Ramanujan birthday celebrations of their math club next Saturday evening. Those who attended the meeting found themselves hooked to the programme. No wonder it became the talk of the campus among primary school children for a week.

base 2 Table

Index	Outcome	Multiplication through addition:		
0	1	$16 \times 32 = 512$	$16 \times 32 \times 1024 = 524288$	$256 \times 512 = 131072$
1	2	$4 + 5 = 9$	$4 + 5 + 10 = 19$	$8 + 9 = 17$
2	4	Division through Subtraction:		
3	8			
4	16	Fractional index into multiplication:		
5	32			
6	64	$512 \div 64 = 8$	$2048 \div 128 = 16$	
7	128	$9 - 6 = 3$	$11 - 7 = 4$	
8	256	The math master complimented all the contributors at the meeting and observed :		
9	512			
10	1024	$128^2 = 128 \times 128 = 16384$	$64^3 = 26244$	$\sqrt{256}$ or $256^{1/2}$
11	2048	$7 + 7 = 14 = 2 \times 7$	$3 \times 6 = 18$	$\frac{1}{2} \times 8 = 4$
12	4096	Once the table is available it can be used in doing multiplication and division, raising to a power and finding its root (addition and subtraction cannot of course be done) . The table is called the logarithmic table, the index receiving the special name logarithm.		
13	8192			
14	16384			
15	32768			
16	65536			
17	131072			
18	262144			
19	524288			
20	1048576			

Can you show $\log_2 8 = \log_8 8^3$? Build base 10 log table and see what it results.

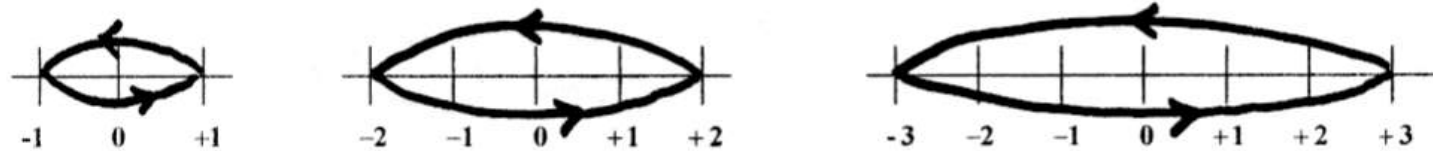
Can (-1) have square root?

The trio sat under the banyan tree and reviewed about the talks and events in the math club. John said, 'I am curious about $\sqrt{-1}$. Does it exist? Ravi said, 'we know $\sqrt{4}$. It is +2 or -2. What could be $\sqrt{-4}$? It cannot be -2 or +2 as $(-2)^2 = (2)^2 = 4$.

Salim said, 'Let us take the root of negative numbers, one by one. $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-4}$, etc..

John said, ' $\sqrt{-2} = \sqrt{2 \times -1} = \sqrt{2} \cdot \sqrt{-1}$, $\sqrt{-3} = \sqrt{3 \times -1} = \sqrt{3} \cdot \sqrt{-1}$, $\sqrt{-4} = \sqrt{4 \times -1} = \sqrt{4} \cdot \sqrt{-1} = 2 \cdot \sqrt{-1}$ and so on. So we must know what square root of -1 means'. They found themselves blocked and so they met their math master and explained to him their problem.

The master said that this was faced by many eminent mathematicians. **Euler** one of the most creative and productive mathematicians of the 18th century said, 'It is not real. It is **imaginary**. So let it be written 'i' for convenience, making it a new unit. Soon it was found that it is **connected with angle**. I shall give you a sequence of diagrams. Observe and discover the connections.



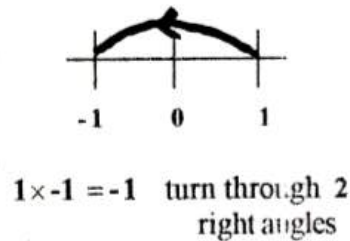
Ravi found out the connection first. His working was as follows:

$1 \times -1 = -1$	turn through 2 right angles
$-1 \times -1 = 1$	turn through 2 right angles again
$2 \times -1 = -2$	turn through 2 right angles again
$-2 \times -1 = 2$	turn through 2 right angles again

John said, 'Multiplication by -1 is turn through 2 right angles. Multiplying again by -1 means turn through 2 right angles more. So repeated multiplication by -1 is connected to repeated addition of 2 right angles.'

Nick name becomes a pet name

The trio thought about 'i' before going to bed. They had a collective discussion after morning assembly. After breakfast, they prepared a paper for being put up in the bulletin board at the dining hall the next day.



$-1 \times -1 = 1$ turn again through 2 right angles



$1 \times (-1)^2 = 1$ turn through $2 \times (2 \text{ right angles})$

Change in position
(multiplication by)

-1
 $(-1)^2$
 $(-1)^3$
 $(-1)^4$

Change in amount of turn

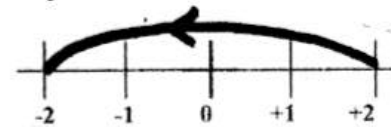
2 rt angles
4 rt angles
6 rt angles
8 rt angles

Salim asked, 'what about turn through 1 rt angle?'. 'That is what we have to fix.', John replied and started drawing through right angles.

x^2 and -1 means the same. So $x^2 = -1$ giving

$x = \sqrt{-1}$ or i . $x^3 = x^2 \cdot x = -1 \times i = -i$.

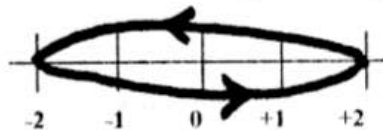
$x^4 = x^2 \cdot x^2 = -1 \times -1 = +1$. So there is nothing imaginary about 'i'. 'i' means turn thru 90° .



$2 \times -1 = 2$ turn through 2 right angles



$-2 \times -1 = 2$ turn again through 2 right angles



$2 \cdot (-1)^2 = 2$ turn through 2 (2 right angles)

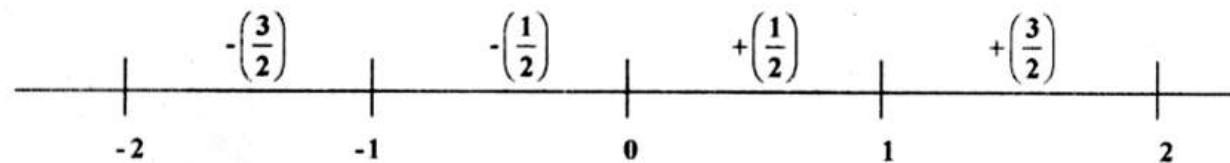
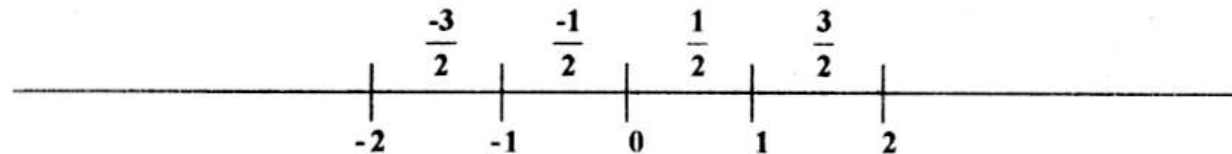
i.e. 4 rt angles

Multiplication by i i^2 i^3 i^4 i^5

Change in angle 1 rt angle 2 rt angles 3 rt angles 4 rt angles 5 rt angles

How come $\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$?

John borrowed a book from the library. When he browsed it, he saw the P.L. $\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$. He became curious and wanted to find out how to obtain it. He scribbled and could see the emerging pattern. His jottings are given below.



$\frac{-4}{+2}$	$\frac{-3}{+2}$	$\frac{-2}{+2}$	$\frac{-1}{+2}$	$\frac{0}{+2}$	$\frac{+1}{+2}$	$\frac{+2}{+2}$	$\frac{+3}{+2}$	$\frac{+4}{+2}$
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$\frac{+4}{-2}$	$\frac{+3}{-2}$	$\frac{+2}{-2}$	$\frac{+1}{-2}$	$\frac{0}{-2}$	$\frac{-1}{-2}$	$\frac{-2}{-2}$	$\frac{-3}{-2}$	$\frac{-4}{-2}$
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P.L. $-\left(\frac{a}{b}\right) = \frac{-a}{+b} = \frac{+a}{-b}$

He showed the jottings to his friends Ravi and Salim. They asked him, 'What provoked you to think of this?'. John shown the book. Ravi and Salim saw, 'Congrats. Let us go and meet our math master.'

Adding without adding

John's father subscribed for his *Junior Mathematician* (Published *thrice* a year by the 'Association of Mathematics Teachers of India' Ph. no: 844 1523). In the bumper issue released to mark the commencement of summer vacation, he read some articles which fascinated him.

The behaviour of 2 and its patterns in continued addition improve his fancy.

$$1 = 1 = 2 - 1 = 2^1 - 1$$

$$1 + 2 = 3 = 4 - 1 = 2^2 - 1$$

$$1 + 2 + 4 = 7 = 8 - 1 = 2^3 - 1$$

$$1 + 2 + 4 + 8 = 15 = 16 - 1 = 2^4 - 1$$

$$1 + 2 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 1$$

He enquired about adding the reciprocals of the powers of 2 and was delighted to discover the pattern.

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8}$$

$$\frac{1}{2} + \frac{1}{2^2} = 1 - \frac{1}{2^2}$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 1 - \frac{1}{2^3}$$

$$\text{P.L.} \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

He went to see the math master. But the master was busy discussing with student groups from higher classes. He decided to meet the master later.

(1a)